$e^+, e^-$ annihilation

$(e^+, e^-)$ annihilation
**$(e^+, e^-)$ annihilation into two photons**

$$e^+ + e^- \rightarrow \gamma + \gamma$$

(need two $\gamma$ for momentum conservation, if the $e^-$ is assumed to be free).

Theoretically, $(e^+, e^-)$ annihilation is related to Compton scattering by crossing symmetry:

- incoming $e^+ \leftrightarrow$ outgoing $e^-$
- outgoing $\gamma \leftrightarrow$ incoming $\gamma$
total cross section per atom

The cross-section formula of Heitler is used [Heitl54]:

\[ \sigma(Z, E) = \frac{Z\pi r_e^2}{\gamma + 1} \left[ \frac{\gamma^2 + 4\gamma + 1}{\gamma^2 - 1} \ln \left( \gamma + \sqrt{\gamma^2 - 1} \right) - \frac{\gamma + 3}{\sqrt{\gamma^2 - 1}} \right] \]

\[ E = \text{total energy of the incident positron} \]

\[ \gamma = \frac{E}{mc^2} \]

\[ r_e = \text{classical electron radius} \]

The cross section decreases with increasing E.

The nonrelativistic limit is:

\[ \sigma_{nr}(Z, E) \sim \frac{Z\pi r_e^2}{\beta} \]
Mean free path

\[ \lambda(E) = \left( \sum_i n_{ati} \cdot \sigma(Z_i, E) \right)^{-1} \]

\( n_{ati} \): nb of atoms per volume of the \( i^{th} \) element in the material.

At initialization stage, the function `BuildPhysicsTables()` computes and tabulates:

- `crossSectionPerAtom` for all elements
- `meanFreePath` for all materials
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number of interactions per cm in Aluminium

Tables for POSITRON in Aluminium

ANNI X-sec (1/cm)

positron kinetic energy (GeV)
**kinematical limits**

\[ e^+ e^- \rightarrow \gamma_a \gamma_b \]

The incident \( e^+ \) has a total energy: \( E = T + mc^2 \)

The total available energy is: \( E_{tot} = E + mc^2 = E_a + E_b \)

Let define the ratio of energy transferred to one photon (says \( \gamma_a \)) :

\[ \epsilon = \frac{E_a}{E_{tot}} = \frac{E_a}{T + 2mc^2} \]

Energy-momentum conservation gives :

\[ \epsilon_{\text{min}} = \frac{E_a^{\text{min}}}{E_{tot}} = \frac{1}{2} \left[ 1 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} \right] \quad \epsilon_{\text{max}} = \frac{E_a^{\text{max}}}{E_{tot}} = \frac{1}{2} \left[ 1 + \sqrt{\frac{\gamma - 1}{\gamma + 1}} \right] \]

Therefore the range of \( \epsilon \) is:

\[ \epsilon \in [\epsilon_{\text{min}}, \epsilon_{\text{max}}] \equiv [\epsilon_{\text{min}}, 1 - \epsilon_{\text{min}}] \]
**sample the gamma energy**

The differential cross-section is:

\[
\frac{d\sigma(Z, \epsilon)}{d\epsilon} = \frac{Z\pi r_e^2}{\gamma - 1} \frac{1}{\epsilon} \left[ 1 + \frac{2\gamma}{(\gamma + 1)^2} - \epsilon - \frac{1}{(\gamma + 1)^2} \frac{1}{\epsilon} \right]
\]

The formula can be factorized:

\[
\frac{d\sigma(Z, \epsilon)}{d\epsilon} = \frac{Z\pi r_e^2}{\gamma - 1} N f(\epsilon) g(\epsilon)
\]

where:

\[
N = \ln(\epsilon_{\text{max}}/\epsilon_{\text{min}})
\]
\[
f(\epsilon) = \frac{1}{N \epsilon}
\]
\[
g(\epsilon) = \left[ 1 + \frac{2\gamma}{(\gamma + 1)^2} - \epsilon - \frac{1}{(\gamma + 1)^2} \frac{1}{\epsilon} \right] \equiv 1 - \epsilon + \frac{2\gamma \epsilon - 1}{\epsilon(\gamma + 1)^2}
\]
Given 2 random numbers $r_a, r_b \in [0, 1]$:

1. sample $\epsilon$ from $f(\epsilon) : \epsilon = \epsilon_{\min} \left[ \frac{\epsilon_{\max}}{\epsilon_{\min}} \right] r_a$

2. reject $\epsilon$ if $g(\epsilon) < r_b$

Then the photon energies are:

$$E_a = \epsilon \: E_{\text{tot}} \quad E_b = (1 - \epsilon) \: E_{\text{tot}}$$

The function `PostStepDoIt()` sample $\epsilon$ and compute the final kinematic
compute the final kinematic

Lets be \( \theta \) the angle between the incident \( e^+ \) and \( \gamma_a \).

From the energy-momentum conservation:

\[
\cos \theta = \frac{1}{P c} \left[ T + mc^2 \frac{2\epsilon - 1}{\epsilon} \right] = \frac{\epsilon(\gamma + 1) - 1}{\epsilon \sqrt{\gamma^2 - 1}}
\]

The azimuthal angle, \( \phi \), is generated isotropically.

The momentum vector of the photons, \( \mathbf{P}_{\gamma_a} \) and \( \mathbf{P}_{\gamma_b} \) are computed from the energy-momentum conservation and transformed to the global coordinate system.
The annihilation in fly is not the dominant process. Most of the time the positron comes at rest and does a positronium with the electron.

The positronium decays in two-photon (in 0.125 nanosec) or three-photon state (in 142 nanosec.)

The function AtRestDoIt treats this case. It generates two photons with energy $E_{\gamma} = mc^2$. The angular distribution is isotropic.

The $(e^+, e^-)$ can also annihilate in a single photon: the other photon is absorbed by the recoil nucleus. However this mechanism is suppressed by a factor $\alpha^4$. 
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$e^+$ 30 MeV in 10 cm Aluminium. Annihilation in fly (left), at rest (right).
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References